

Contents lists available at [ScienceDirect](http://ScienceDirect.com)

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijssolstr

An exact solution for a multilayered two-dimensional decagonal quasicrystal plate

Lian-Zhi Yang^{a,b}, Yang Gao^{a,*}, Ernian Pan^c, Natalie Waksamanski^c^a College of Science, China Agricultural University, Beijing 100083, China^b College of Engineering, China Agricultural University, Beijing 100083, China^c Department of Civil Engineering, University of Akron, Akron, OH 44325-3905, USA

ARTICLE INFO

Article history:

Received 10 October 2013

Received in revised form 2 January 2014

Available online 27 January 2014

Keywords:

Two-dimensional quasicrystals

Simply supported

Homogeneous

Multilayered

Plate

ABSTRACT

By extending the pseudo-Stroh formalism to two-dimensional decagonal quasicrystals, an exact closed-form solution for a simply supported and multilayered two-dimensional decagonal quasicrystal plate is derived in this paper. Based on the different relations between the periodic direction and the coordinate system of the plate, three internal structure cases for the two-dimensional quasicrystal layer are considered. The propagator matrix method is also introduced in order to treat efficiently and accurately the multilayered cases. The obtained exact closed-form solution has a concise and elegant expression. Two homogeneous quasicrystal plates and a sandwich plate made of a two-dimensional quasicrystal and a crystal with two stacking sequences are investigated using the derived solution. Numerical results show that the differences of the periodic direction have strong influences on the stress and displacement components in the phonon and phason fields; different coupling constants between the phonon and phason fields will also cause differences in physical quantities; the stacking sequences of the multilayer plates can substantially influence all physical quantities. The exact closed-form solution should be of interest to the design of the two-dimensional quasicrystal homogeneous and laminated plates. The numerical results can also be employed to verify the accuracy of the solution by numerical methods, such as the finite element and difference methods, when analyzing laminated composites made of quasicrystals.

© 2014 Elsevier Ltd. All rights reserved.

1. Introduction

Quasicrystals (QCs) emerged as a new structure of solid matter based from a diffraction image of rapidly cooled Al-Mn alloys around 1982 (Shechtman et al., 1984; Levine and Steinhardt, 1984). The discovery was revolutionary because QCs are contrary to conventional crystals in that they lack translation symmetry. Among approximately 200 individual QCs observed to date, two-dimensional (2D) QCs with fine thermal stability play an important role in this kind of matter (Fan, 2011). A 2D QC is defined as a three-dimensional (3D) body where its atomic arrangement is quasi-periodic in a plane and periodic along the direction normal to the plane. To describe the linear elastic mechanical behaviors of the material at room temperature (Fan, 2011), the generalized linear elastic theory of QCs based on the notion of continuum mechanics was established by Ding et al. (1993). Based on the symmetry breaking principle of Landau, the physical basis of elasticity of QCs is formulated by Bak (1985a, 1985b) and Levine and Steinhardt (1984). In this theory, the phonon displacement field

is analogous to the displacement field of traditional continuum mechanics which describes shape and volume changes of unit cells. Additional degrees of freedom are introduced as the phason displacement field attributing to the quasi-periodic lattice structure in QCs. The phason displacement field corresponds to atomic rearrangement of unit cells. Due to elementary excitation, the phonon mode is propagating whereas the phason mode is diffusive. Recent reviews on the linear elastic theory of QCs can be found in Hu et al. (2000) and Fan (2011, 2013).

Due to their low friction coefficient, high hardness, low adhesion, high wear resistance and low level of porosity, QCs are predominantly used in industry as coatings or thin films of metals (Balbyshev et al., 2004). Studies in QC multilayered plates offer guidance in understanding the stresses and deformations of QC coatings or films. For crystal composites, analytical solutions for simply supported plates have been obtained (Noor and Burton, 1990; Pan, 2001). Although three point bending solution for QC plate under static and transient dynamic loads has been obtained (Sladek et al., 2013), it was for one-dimensional QCs. The complexity of the QC basic equations of elasticity increases considerably from 1D QC to 2D QC which limits most of studies on 2D QCs to the defect problems in infinite spaces (Zhou and Fan, 2001;

* Corresponding author. Tel.: +86 10 6273 6411.

E-mail address: gaoyang@gmail.com (Y. Gao).

Fan et al., 2004). Up to now, no exact closed-form solution for mechanical problems of plates in finite space has been reported in literature for 3D problems of 2D QC composites.

In this paper, we derive an exact closed-form solution for a multilayered 2D decagonal QC plate under surface loadings with simply supported lateral boundaries. The powerful pseudo-Stroh formalism (Pan, 2001) is first extended to 2D QCs to obtain the general solution for each homogeneous QC layer. Based on the different relations between the periodic direction and the coordinate system of the plate, different internal structure cases for the 2D QC layer are considered. Furthermore, a multilayered plate containing both QC layers and crystal layers as a special case is investigated in details with the propagator matrix method (Pan, 1997a) being introduced to treat the corresponding multilayered cases. As numerical illustrations, three examples are discussed.

2. Basic equations

Consider a 2D QC with x_1 and x_2 as the quasi-periodic directions and x_3 as the periodic direction referring to a rectangular Cartesian coordinate system (x_1, x_2, x_3) . The phason displacements w_m ($m = 1, 2$) exist in addition to phonon displacements u_i ($i = 1, 2, 3$). Phonon displacements correspond to the translation of atoms, whereas phason displacements correspond to the rearrangement of atoms. According to the linear elastic theory of QCs (Ding et al., 1993), the strain–displacement relations for 2D QCs are given by

$$\varepsilon_{ij} = (\partial_j u_i + \partial_i u_j)/2, \quad w_{mj} = \partial_j w_m, \quad (1)$$

where $j = 1, 2, 3$, $\partial_j = \partial/\partial x_j$, ε_{ij} and w_{mj} denote the phonon and phason strains, respectively.

In the absence of body forces, the static equilibrium equations are

$$\partial_j \sigma_{ij} = 0, \quad \partial_j H_{mj} = 0, \quad (2)$$

where σ_{ij} and H_{mj} respectively denote the phonon and phason stresses, and repeated indices imply the summation from 1 to 3. Ding et al. (1993) derived the equilibrium equation from the law of momentum conservation. It should be noted that it is possible to write equilibrium equations for a generalized degree of freedom in the form of the second Newton's law only if there exists a corresponding conservation law. As such, although the phason mode in QCs corresponds to atomic jumps or diffusion, there is no conservation law corresponding to the diffusion of atoms (Rochal and Lorman, 2002).

We arrange the strain components in phonon and phason fields respectively in two vectors as

$$\{\bar{\gamma}\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\}^t, \quad (3)$$

$$\{\bar{w}\} = \{w_{11}, w_{22}, w_{23}, w_{12}, w_{13}, w_{21}\}^t,$$

in which the superscript “ t ” represents the transpose, $\gamma_{ij} = 2\varepsilon_{ij}$ ($i \neq j$), and the stress components are ordered similarly as

$$\{\bar{\sigma}\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\}^t, \quad (4)$$

$$\{\bar{H}\} = \{H_{11}, H_{22}, H_{23}, H_{12}, H_{13}, H_{21}\}^t.$$

Making use of the displacement and stress vectors in Eqs. (3) and (4), the linear constitutive equations of 2D QCs can be expressed by the following form (Fan, 2011; Ding et al., 1993):

$$\bar{\sigma}_k = C_{kl} \bar{\gamma}_l + R_{kl} \bar{w}_l, \quad (5)$$

$$\bar{H}_k = R_{lk} \bar{\gamma}_l + K_{kl} \bar{w}_l,$$

where $k, l = 1, 2, \dots, 6$, C_{kl} and K_{kl} are, respectively, the elastic constants in phonon and phason fields, R_{kl} are the coupling constants between the phonon and phason fields. For 2D decagonal QCs with the point groups $10mm$, $10\bar{2}2$, $10/m2$, $10/m\bar{3}m$, the three constant tensors in Eq. (5) can be written as

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \quad (6)$$

$$\mathbf{R} = \begin{bmatrix} R_1 & R_1 & 0 & 0 & 0 & 0 \\ -R_1 & -R_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_1 & 0 & R_1 \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} K_1 & K_2 & 0 & 0 & 0 & 0 \\ K_2 & K_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_1 & 0 & -K_2 \\ 0 & 0 & 0 & 0 & K_4 & 0 \\ 0 & 0 & 0 & -K_2 & 0 & K_1 \end{bmatrix}.$$

Although the fundamental equations have been presented in differential form, they can also be expressed in variational form by introducing an energy functional for quasicrystals (Fan, 2011; Altay and Dökmeci, 2012; Shi, 2005).

3. Problem description and general solution

Consider a multilayered 2D decagonal QC plate as shown in Fig. 1 with horizontal dimensions $x \times y = L_x \times L_y$ and a total thickness $z = H$ in a rectangular Cartesian coordinate system (x, y, z) with its four sides being simply supported. Let j denote the j -th layer of the layered plate. For layer j , its lower and upper interfaces are defined, respectively, as z_j and z_{j+1} . Thus, for an N -layered plate with total thickness H , it is clear that $z_1 = 0$ and $z_{N+1} = H$. Along the interfaces of the layers, the displacements and z -direction traction stresses are assumed to be continuous, i.e.

$$\begin{cases} (u_i)_j = (u_i)_{j+1}, & (w_m)_j = (w_m)_{j+1}, \\ (\sigma_{iz})_j = (\sigma_{iz})_{j+1}, & (H_{mz})_j = (H_{mz})_{j+1}, \end{cases} \quad \text{at the interface of layer } j \text{ and } j+1. \quad (7)$$

The coordinate system (x, y, z) in Fig. 1 is a global one and it is independent of the materials of the plate. We also induce the local material coordinate system (x_1, x_2, x_3) mentioned in Section 2 which characterizes the physical properties of the QC layer. Both the origins of the global and local coordinate systems O and O' are at one of the four corners on the bottom surface with the same position. According to the relative orientation of the local material coordinate system with respect to the global coordinates, three cases of internal structures of the 2D QC plate are investigated. As a special case, a multilayered plate containing both QC layers and crystal layers will be considered.

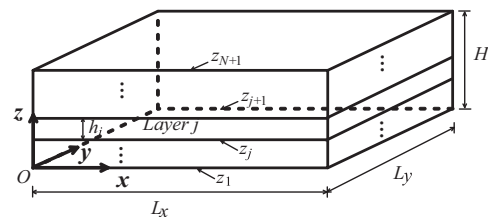


Fig. 1. A multilayered 2D QC plate.

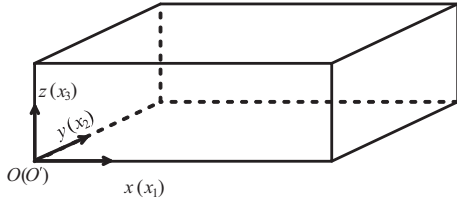


Fig. 2. A 2D QC homogenous plate of Case 1.

Case 1. We assume that the global and local coordinate systems having the relation $(x, y, z) = (x_1, x_2, x_3)$ as shown in Fig. 2. Accordingly, the periodic direction of the 2D QC is the z -direction or the thickness direction of the plate.

The solution of the displacement vector of the homogenous 2D QC plate is assumed to take the following form:

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \\ w_x \\ w_y \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \end{Bmatrix} = e^{sz} \begin{Bmatrix} a_1 \cos px \sin qy \\ a_2 \sin px \cos qy \\ a_3 \sin px \sin qy \\ a_4 \cos px \sin qy \\ a_5 \sin px \cos qy \end{Bmatrix}, \quad (8)$$

where

$$p = n\pi/L_x, \quad q = m\pi/L_y, \quad (9)$$

with n and m being two positive integers, and the coefficients to be determined are s, a_1, a_2, a_3, a_4 and a_5 . It can be seen that the displacement vector satisfies the simply supported displacement boundary conditions:

$$\begin{aligned} x = 0 \text{ and } L_x : u_y = u_z = w_y = 0; \\ y = 0 \text{ and } L_y : u_x = u_z = w_x = 0. \end{aligned} \quad (10)$$

It is noted that the solution in Eq. (8) represents only one of the terms in a double Fourier series expansion when solving a general boundary value problem. Therefore, in general, summations for n and m over suitable ranges are implied whenever the sinusoidal term appears.

Substituting Eq. (8) into the constitutive Eq. (5), the z -direction traction vector can be written as

$$\mathbf{t} = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \\ H_{xz} \\ H_{yz} \end{Bmatrix} = \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \\ H_{13} \\ H_{23} \end{Bmatrix} = e^{sz} \begin{Bmatrix} b_1 \cos px \sin qy \\ b_2 \sin px \cos qy \\ b_3 \sin px \sin qy \\ b_4 \cos px \sin qy \\ b_5 \sin px \cos qy \end{Bmatrix}. \quad (11)$$

$$\mathbf{Q} = \begin{bmatrix} -(C_{11}p^2 + C_{66}q^2) & -pq(C_{12} + C_{66}) & 0 & R_1(q^2 - p^2) & -2R_1pq \\ -pq(C_{12} + C_{66}) & -(C_{66}p^2 + C_{11}q^2) & 0 & 2R_1pq & R_1(q^2 - p^2) \\ 0 & 0 & -C_{44}(p^2 + q^2) & 0 & 0 \\ R_1(q^2 - p^2) & 2R_1pq & 0 & -K_1(p^2 + q^2) & 0 \\ -2R_1pq & R_1(q^2 - p^2) & 0 & 0 & -K_1(p^2 + q^2) \end{bmatrix}. \quad (19)$$

The two vectors

$$\mathbf{a} = \{a_1, a_2, a_3, a_4, a_5\}^t, \quad \mathbf{b} = \{b_1, b_2, b_3, b_4, b_5\}^t, \quad (12)$$

are introduced to represent the coefficients in Eqs. (8) and (11), respectively. By using the constitutive Eq. (5), the vectors \mathbf{b} and \mathbf{a} have the following relation:

$$\mathbf{b} = (-\mathbf{P}^t + s\mathbf{T})\mathbf{a}, \quad (13)$$

in which

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & C_{13}p & 0 & 0 \\ 0 & 0 & C_{13}q & 0 & 0 \\ -C_{44}p & -C_{44}q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} C_{44} & 0 & 0 & 0 & 0 \\ 0 & C_{44} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & K_4 & 0 \\ 0 & 0 & 0 & 0 & K_4 \end{bmatrix}. \quad (14)$$

Similarly, the other stress components in Eq. (4) are obtained as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \\ H_{xx} \\ H_{yy} \\ H_{xy} \\ H_{yx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \\ H_{11} \\ H_{22} \\ H_{12} \\ H_{21} \end{Bmatrix} = e^{sz} \begin{Bmatrix} c_1 \sin px \sin qy \\ c_2 \cos px \cos qy \\ c_3 \sin px \sin qy \\ c_4 \sin px \sin qy \\ c_5 \sin px \sin qy \\ c_6 \cos px \cos qy \\ c_7 \cos px \cos qy \end{Bmatrix}, \quad (15)$$

where

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{Bmatrix} = \begin{bmatrix} -C_{11}p & -C_{12}q & C_{13}s & -R_1p & -R_1q \\ C_{66}q & C_{66}p & 0 & -R_1q & R_1p \\ -C_{12}p & -C_{11}q & C_{13}s & R_1p & R_1q \\ -R_1p & R_1q & 0 & -K_1p & -K_2q \\ -R_1p & R_1q & 0 & -K_2p & -K_1q \\ -R_1q & -R_1p & 0 & K_1q & -K_2p \\ R_1q & R_1p & 0 & -K_2q & K_1p \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}. \quad (16)$$

By substituting all stress components in Eqs. (11) and (15) into the equilibrium Eq. (2), the following relations are obtained:

$$\begin{aligned} &-(C_{11}p^2 + C_{66}q^2)a_1 - (C_{12}pq + C_{66}pq)a_2 + R_1(q^2 - p^2)a_4 \\ &- 2R_1pqa_5 + s(C_{13}p + C_{44}p)a_3 + s^2C_{44}a_1 = 0, \\ &-(C_{66}pq + C_{12}pq)a_1 - (C_{66}p^2 + C_{11}q^2)a_2 + 2R_1pqa_4 + R_1(q^2 - p^2)a_5 \\ &+ s(C_{13}q + C_{44}q)a_3 + s^2C_{44}a_2 = 0, \\ &-(C_{44}p^2 + C_{44}q^2)a_3 + s(-C_{44}pa_1 - C_{44}qa_2 - C_{13}pa_1 - C_{13}qa_2) + s^2C_{33}a_3 = 0, \\ &R_1(q^2 - p^2)a_1 + 2R_1pqa_2 - (K_1p^2 + K_1q^2)a_4 + s^2K_4a_4 = 0, \\ &- 2R_1pqa_1 + R_1(q^2 - p^2)a_2 - (K_1p^2 + K_1q^2)a_5 + s^2K_4a_5 = 0. \end{aligned} \quad (17)$$

In terms of vector \mathbf{a} , Eq. (17) simplifies to

$$[\mathbf{Q} + s(\mathbf{P} + \mathbf{P}') + s^2\mathbf{T}]\mathbf{a} = \mathbf{0}, \quad (18)$$

where $\mathbf{P}' = -\mathbf{P}^t$, and

It should be noted that Eq. (18) is similar to the Stroh formalism (Stroh, 1958). Thus, this formalism can be appropriately named as the pseudo-Stroh formalism (Pan, 2001).

Case 2. In this case, the local and global coordinate systems have the relation $(x_3, x_1, x_2) = (x, y, z)$ as shown in Fig. 3. The periodic direction of the 2D QC is in the in-plane x -direction of the plate.

The solution of the displacement vector for this case is assumed to take the form as

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \\ w_y \\ w_z \end{Bmatrix} = \begin{Bmatrix} u_3 \\ u_1 \\ u_2 \\ w_1 \\ w_2 \end{Bmatrix} = e^{sz} \begin{Bmatrix} a_1 \cos px \sin qy \\ a_2 \sin px \cos qy \\ a_3 \sin px \sin qy \\ a_4 \sin px \cos qy \\ a_5 \sin px \sin qy \end{Bmatrix}, \quad (20)$$

which satisfies the simply supported boundary conditions of the plate. Substituting Eq. (20) into the constitutive Eq. (5), the following traction vector can be obtained as:

$$\mathbf{Q} = \begin{bmatrix} -(C_{33}p^2 + C_{44}q^2) & -pq(C_{13} + C_{44}) & 0 & 0 & 0 \\ -pq(C_{13} + C_{44}) & -(C_{44}p^2 + C_{11}q^2) & 0 & -R_1q^2 & 0 \\ 0 & 0 & -(C_{66}q^2 + C_{44}p^2) & 0 & -R_1q^2 \\ 0 & -R_1q^2 & 0 & -(K_1q^2 + K_4p^2) & 0 \\ 0 & 0 & -R_1q^2 & 0 & -(K_1q^2 + K_4p^2) \end{bmatrix}. \quad (25)$$

$$\mathbf{t} = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \\ H_{yz} \\ H_{zz} \end{Bmatrix} = \begin{Bmatrix} \sigma_{32} \\ \sigma_{12} \\ \sigma_{22} \\ H_{12} \\ H_{22} \end{Bmatrix} = e^{sz} \begin{Bmatrix} b_1 \cos px \sin qy \\ b_2 \sin px \cos qy \\ b_3 \sin px \sin qy \\ b_4 \sin px \cos qy \\ b_5 \sin px \sin qy \end{Bmatrix}. \quad (21)$$

The relation between \mathbf{b} and \mathbf{a} in Eq. (13) can be shown to remain the same as the previous case, while \mathbf{P} and \mathbf{T} are changed into the following forms:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & C_{13}p & 0 & 0 \\ 0 & 0 & C_{12}q & 0 & R_1q \\ -C_{44}p & -C_{66}q & 0 & R_1q & 0 \\ 0 & 0 & -R_1q & 0 & K_2q \\ 0 & -R_1q & 0 & K_2q & 0 \end{bmatrix}, \quad (22)$$

$$\mathbf{T} = \begin{bmatrix} C_{44} & 0 & 0 & 0 & 0 \\ 0 & C_{66} & 0 & -R_1 & 0 \\ 0 & 0 & C_{11} & 0 & -R_1 \\ 0 & -R_1 & 0 & K_1 & 0 \\ 0 & 0 & -R_1 & 0 & K_1 \end{bmatrix}.$$

The other stress components are obtained as

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xx} \\ H_{yy} \\ H_{yx} \\ H_{zy} \\ H_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{13} \\ \sigma_{33} \\ H_{11} \\ H_{13} \\ H_{21} \\ H_{23} \end{Bmatrix} = e^{sz} \begin{Bmatrix} c_1 \sin px \sin qy \\ c_2 \cos px \cos qy \\ c_3 \sin px \sin qy \\ c_4 \sin px \sin qy \\ c_5 \cos px \cos qy \\ c_6 \sin px \cos qy \\ c_7 \cos px \sin qy \end{Bmatrix}, \quad (23)$$

where

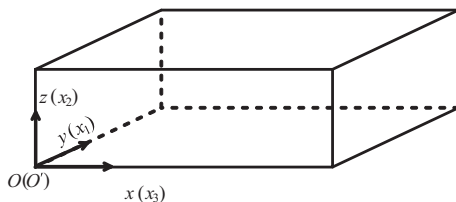


Fig. 3. A 2D QC homogeneous plate of Case 2.

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{Bmatrix} = \begin{bmatrix} -C_{13}p & -C_{11}q & C_{12}s & -R_1q & R_1s \\ C_{44}q & C_{44}p & 0 & 0 & 0 \\ -C_{33}p & -C_{13}q & C_{13}s & 0 & 0 \\ 0 & -R_1q & -R_1s & -K_1q & K_2s \\ 0 & 0 & 0 & K_4p & 0 \\ 0 & R_1s & R_1q & -K_2s & K_1q \\ 0 & 0 & 0 & 0 & K_4p \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}. \quad (24)$$

Substituting all the stress components in Eqs. (21) and (23) into the equilibrium Eq. (2), Eq. (18) remains valid in this case, and the new \mathbf{Q} is obtained as

Case 3. We assume that the local and global coordinate systems to be related by $(x_2, x_3, x_1) = (x, y, z)$, as shown in Fig. 4. In this case, the periodic direction of the 2D QC is parallel to the y axis.

The solution of the displacement vector is assumed to be

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \\ w_x \\ w_z \end{Bmatrix} = \begin{Bmatrix} u_2 \\ u_3 \\ u_1 \\ w_2 \\ w_1 \end{Bmatrix} = e^{sz} \begin{Bmatrix} a_1 \cos px \sin qy \\ a_2 \sin px \cos qy \\ a_3 \sin px \sin qy \\ a_4 \cos px \sin qy \\ a_5 \sin px \sin qy \end{Bmatrix}. \quad (26)$$

Accordingly, the corresponding traction vector is

$$\mathbf{t} = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \\ H_{xz} \\ H_{zz} \end{Bmatrix} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{31} \\ \sigma_{11} \\ H_{21} \\ H_{11} \end{Bmatrix} = e^{sz} \begin{Bmatrix} b_1 \cos px \sin qy \\ b_2 \sin px \cos qy \\ b_3 \sin px \sin qy \\ b_4 \cos px \sin qy \\ b_5 \sin px \sin qy \end{Bmatrix}. \quad (27)$$

The same relation between \mathbf{b} and \mathbf{a} in Eq. (13) still holds in this case, while \mathbf{P} and \mathbf{T} are now rewritten as

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & C_{12}p & 0 & -R_1p \\ 0 & 0 & C_{13}q & 0 & 0 \\ -C_{66}p & -C_{44}q & 0 & -R_1p & 0 \\ 0 & 0 & R_1p & 0 & K_2p \\ R_1p & 0 & 0 & K_2p & 0 \end{bmatrix}, \quad (28)$$

$$\mathbf{T} = \begin{bmatrix} C_{66} & 0 & 0 & R_1 & 0 \\ 0 & C_{44} & 0 & 0 & 0 \\ 0 & 0 & C_{11} & 0 & R_1 \\ R_1 & 0 & 0 & K_1 & 0 \\ 0 & 0 & R_1 & 0 & K_1 \end{bmatrix}.$$

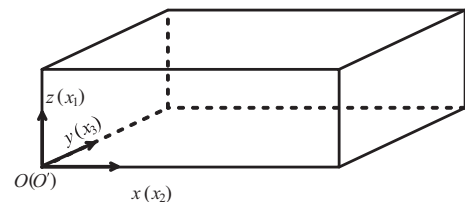


Fig. 4. A 2D homogeneous QC plate of Case 3.

The other stress components are expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \\ H_{xx} \\ H_{xy} \\ H_{zx} \\ H_{zy} \end{Bmatrix} = \begin{Bmatrix} \sigma_{22} \\ \sigma_{23} \\ \sigma_{33} \\ H_{22} \\ H_{23} \\ H_{12} \\ H_{13} \end{Bmatrix} = e^{sz} \begin{Bmatrix} c_1 \sin px \sin qy \\ c_2 \cos px \cos qy \\ c_3 \sin px \sin qy \\ c_4 \sin px \sin qy \\ c_5 \cos px \cos qy \\ c_6 \cos px \sin qy \\ c_7 \sin px \cos qy \end{Bmatrix}, \quad (29)$$

where

$$\mathbf{Q} = \begin{bmatrix} -(C_{11}p^2 + C_{44}q^2) & -pq(C_{13} + C_{44}) & 0 & R_1p^2 & 0 \\ -pq(C_{13} + C_{44}) & -(C_{44}p^2 + C_{33}q^2) & 0 & 0 & 0 \\ 0 & 0 & -(C_{66}p^2 + C_{44}q^2) & 0 & R_1p^2 \\ R_1p^2 & 0 & 0 & -(K_1p^2 + K_4q^2) & 0 \\ 0 & 0 & R_1p^2 & 0 & -(K_1p^2 + K_4q^2) \end{bmatrix}. \quad (31)$$

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{Bmatrix} = \begin{bmatrix} -C_{11}p & -C_{13}q & C_{12}s & R_1p & -R_1s \\ C_{44}q & C_{44}p & 0 & 0 & 0 \\ -C_{13}p & -C_{33}q & C_{13}s & 0 & 0 \\ R_1p & 0 & R_1s & -K_1p & K_2s \\ 0 & 0 & 0 & K_4q & 0 \\ -R_1s & 0 & -R_1p & -K_2s & K_1p \\ 0 & 0 & 0 & 0 & K_4q \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix}. \quad (30)$$

Substituting all the stress components in Eqs. (27) and (29) into the equilibrium Eq. (2), we have the same relation expressed in Eq. (18), but with the \mathbf{Q} for this case being

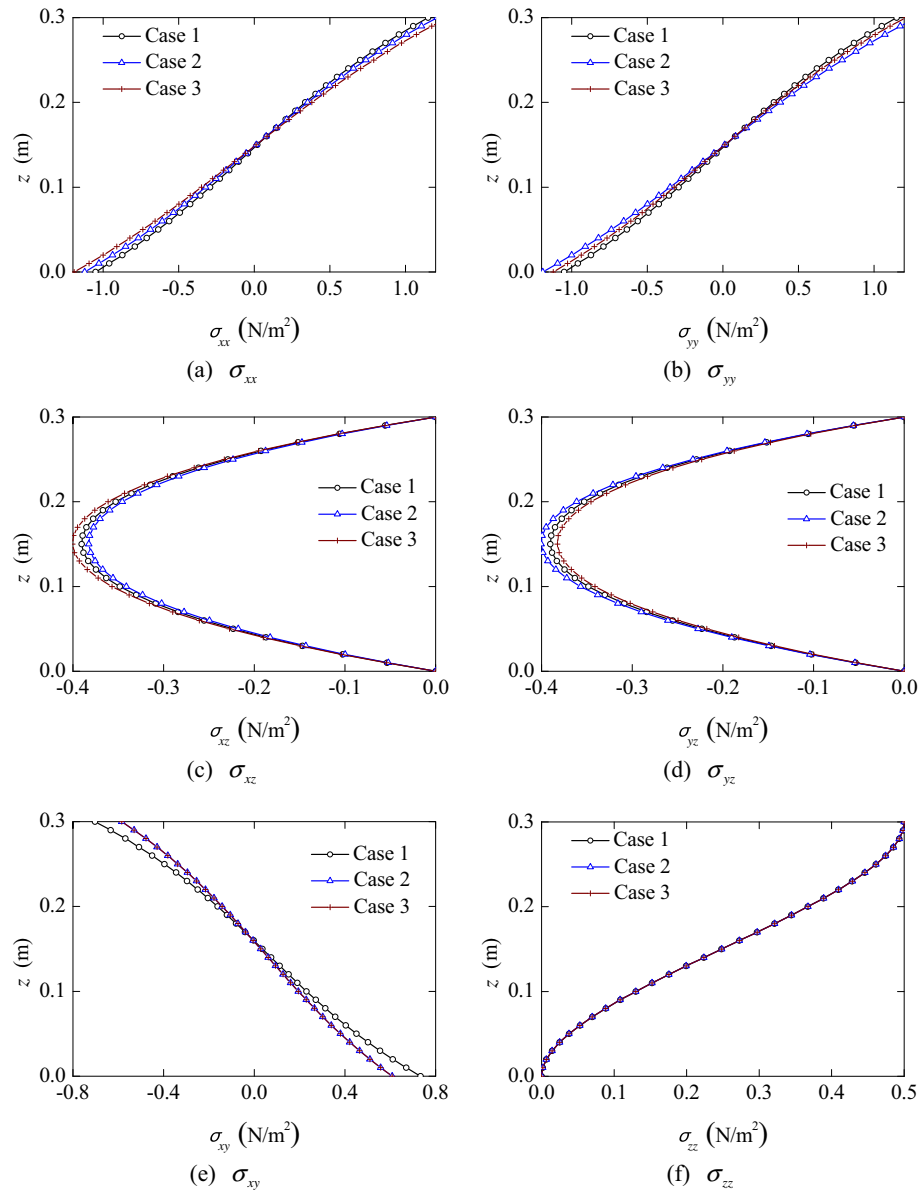


Fig. 5. Variation of the stress components in phonon field along z -direction of the homogeneous plate.

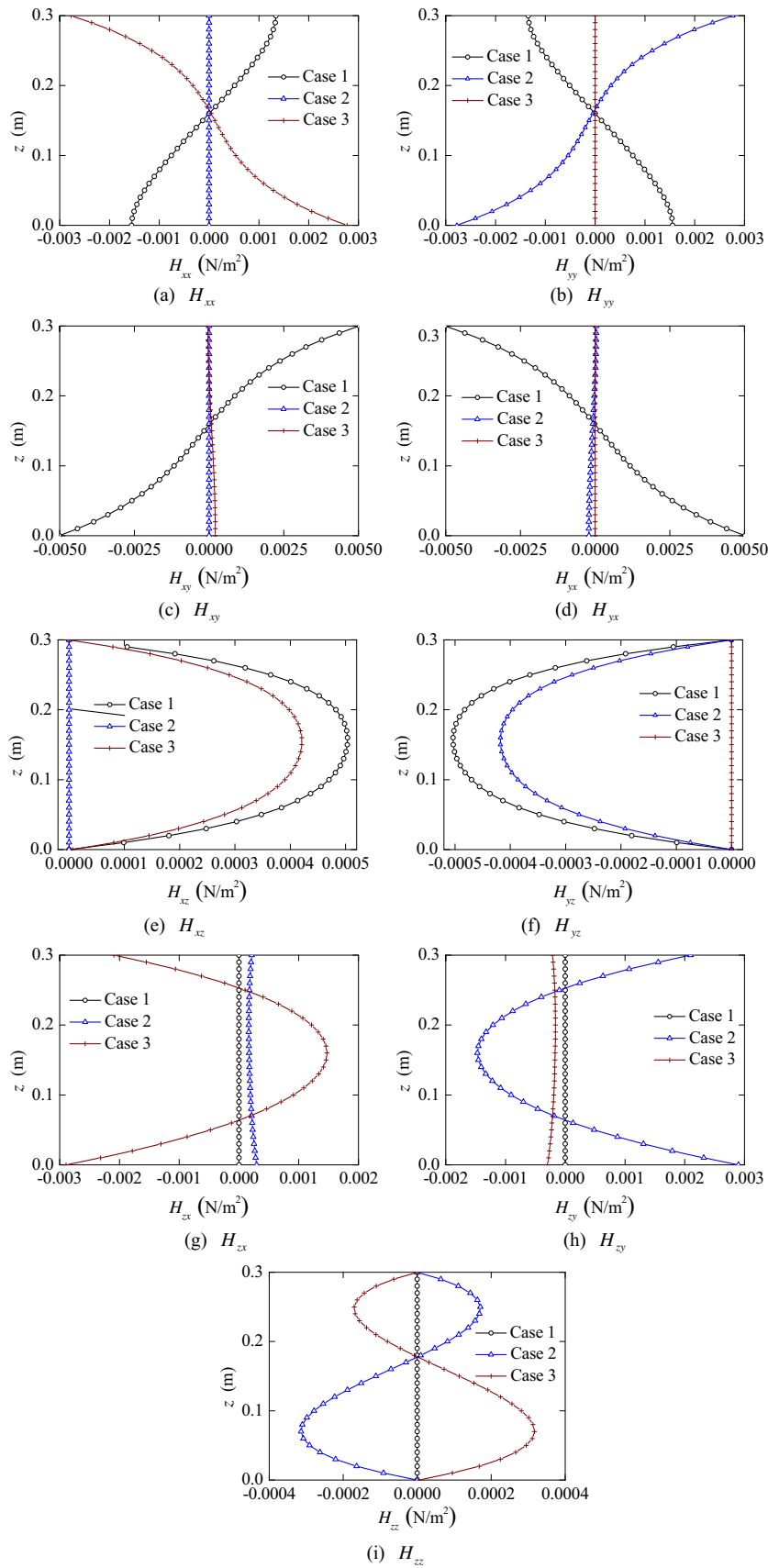


Fig. 6. Variation of the stress components in phason field along z -direction of the homogeneous plate.

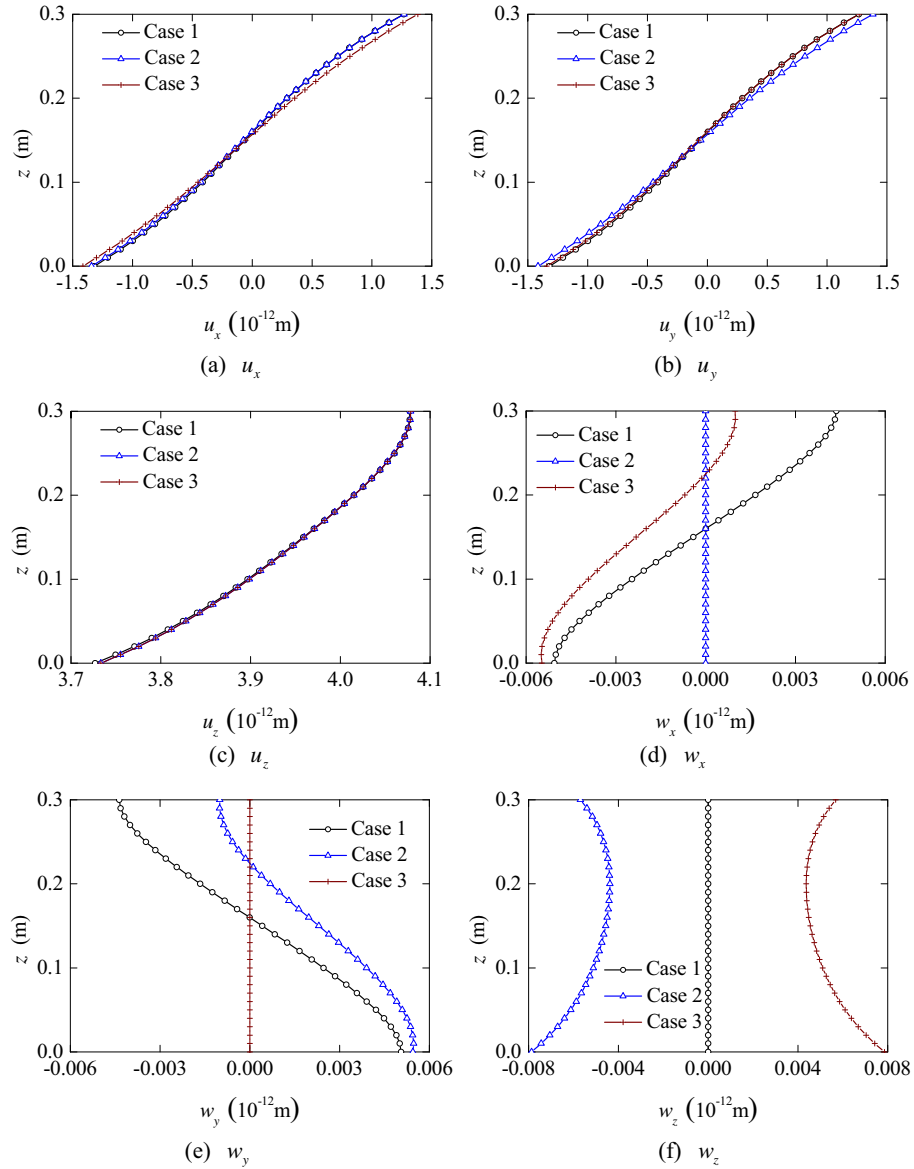


Fig. 7. Variation of the displacement components along z -direction of the homogeneous plate.

From the above analyses, it can be seen that Eqs. (13) and (18) are valid for all the three cases, although the components in the matrices \mathbf{Q} , \mathbf{P} and \mathbf{T} take different forms. The method to deduce the general solution for 2D QC plates is independent of the form of the components taken in the matrices.

Making use of Eqs. (13) and (18), another relation between vectors \mathbf{a} and \mathbf{b} is obtained as

$$\mathbf{b} = -\frac{1}{s}(\mathbf{Q} + s\mathbf{P})\mathbf{a}. \quad (32)$$

Then, by employing Eqs. (13), (32), and (18) can be recast into a 10×10 linear eigensystem

$$\mathbf{N}\boldsymbol{\eta} = s\boldsymbol{\eta}, \quad \boldsymbol{\eta} = \{\mathbf{a}, \mathbf{b}\}^t, \quad (33)$$

where

$$\mathbf{N} = \begin{bmatrix} -\mathbf{T}^{-1}\mathbf{P}' & \mathbf{T}^{-1} \\ -\mathbf{Q} + \mathbf{P}\mathbf{T}^{-1}\mathbf{P}' & -\mathbf{P}\mathbf{T}^{-1} \end{bmatrix}. \quad (34)$$

A nontrivial solution for $\boldsymbol{\eta}$ exists if the determinant of the characteristic matrix in Eq. (33) vanishes. If repeated roots occur, a slight change in the material constants would result in distinct roots with negligible error (Pan, 1997b). Thus, all eigenvalues can be assumed to be distinct. We assume that the first five eigenvalues have positive real parts (if the root is purely imaginary, we then pick up the one with positive imaginary part) and the other five have opposite signs to the first five. The associated eigenvectors \mathbf{a} and \mathbf{b} corresponding to the eigenvalues s follow the same ordering. The first five eigenvectors \mathbf{a} and \mathbf{b} are defined as 5×5 matrices \mathbf{A}_1 and \mathbf{B}_1 , respectively, and the following five eigenvectors are defined as \mathbf{A}_2 and \mathbf{B}_2 . Then the general solution for the displacement vector \mathbf{u} and traction vector \mathbf{t} is derived as

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \langle e^{s^*z} \rangle \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}, \quad (35)$$

where

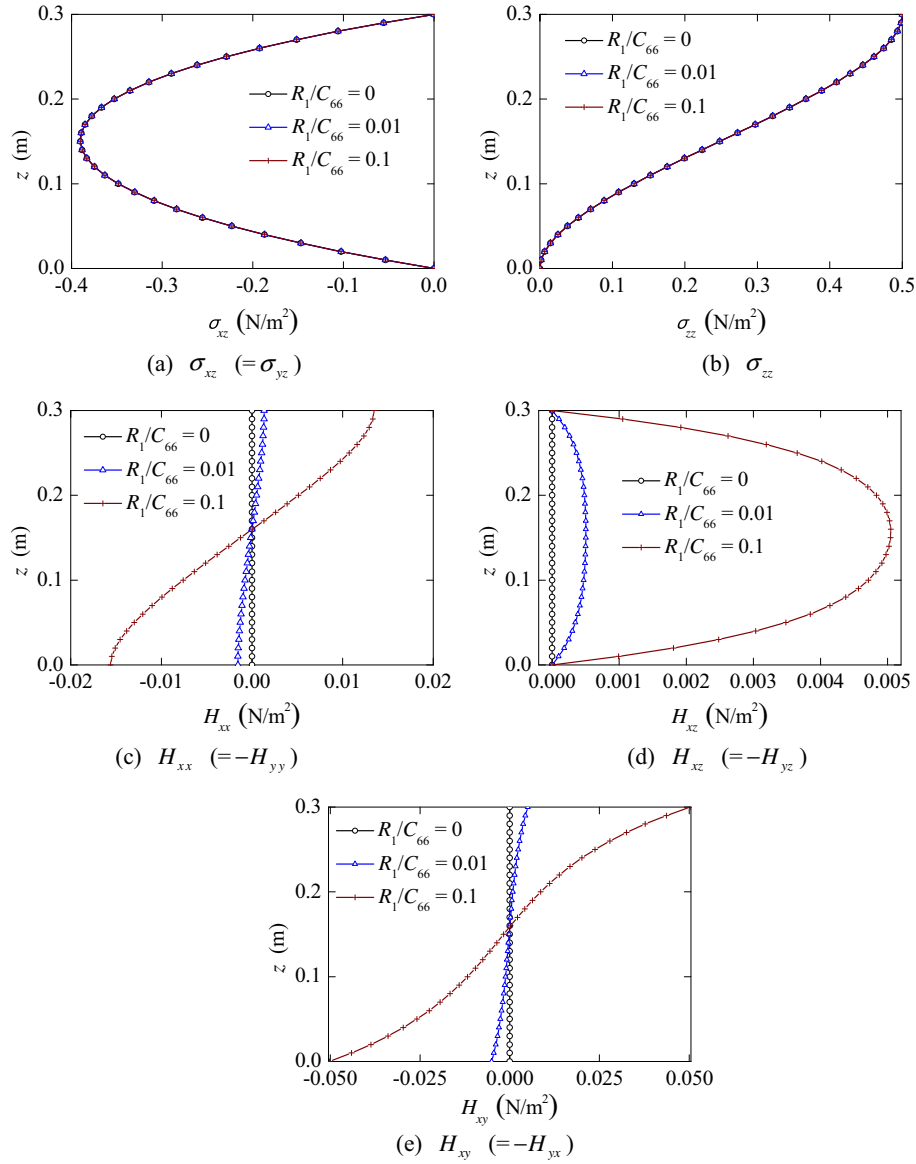


Fig. 8. Variation of the stress components along z -direction of the plate for Case 1 under different coupling constant R_1 .

$$\begin{aligned}
 \mathbf{A}_1 &= [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5], & \mathbf{A}_2 &= [\mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8, \mathbf{a}_9, \mathbf{a}_{10}], \\
 \mathbf{B}_1 &= [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5], & \mathbf{B}_2 &= [\mathbf{b}_6, \mathbf{b}_7, \mathbf{b}_8, \mathbf{b}_9, \mathbf{a}_{10}], \\
 \langle e^{s^z} \rangle &= \text{diag}[e^{s_1 z}, e^{s_2 z}, e^{s_3 z}, e^{s_4 z}, e^{s_5 z}, e^{-s_1 z}, e^{-s_2 z}, e^{-s_3 z}, e^{-s_4 z}, e^{-s_5 z}],
 \end{aligned}
 \quad (36)$$

and \mathbf{K}_1 and \mathbf{K}_2 are two 5×1 constant column matrices to be determined by the boundary conditions of the problem. In Eqs. (33) and (35), the eigenvectors are only related to the material properties of the plates.

The general solution obtained from Eq. (35) is for a homogeneous and simply supported plate consisting of 2D QCs. It should be noted that results of the corresponding thin plate case can be deduced from this solution by expanding the exponential term in terms of a Taylor series (Kausel and Roesset, 1981).

We should point out that crystals can be seen as special QCs with all the phason field physical quantities are zero. In the following, the feasibility of the general solution in Eq. (35) for multilayered plates containing both QC layers and crystal layers is discussed as Case 4. The study is very important in that QCs are always used as films or coatings of crystals in industry.

Case 4. From Eq. (5), it can be seen if we set

$$R_1 \rightarrow 0, \quad K_1 = K_2 = K_4 \rightarrow 0, \quad (37)$$

then

$$\bar{H}_k \rightarrow 0. \quad (38)$$

It can be inferred that, for this limiting case, the phonon stresses and strains of the QC are infinitely close to those in the corresponding purely elastic crystal. Therefore, the general solution in Eq. (35) can be used for the purely elastic crystal simply supported plates by regarding a crystal layer as “a special QC” layer with the phason-field elastic constants satisfying Eq. (37). In other words, the values of the phason elastic constants of crystal layers should not be exactly set to zero, but relatively very small (compared to those in QC layers as discussed further below) to ensure that the system matrices are not singular.

For a multilayered plate containing not only QC layers but also crystal layers, at the interface between QC and crystal, in phason field, only the following boundary condition should be satisfied (Fan et al., 2011):

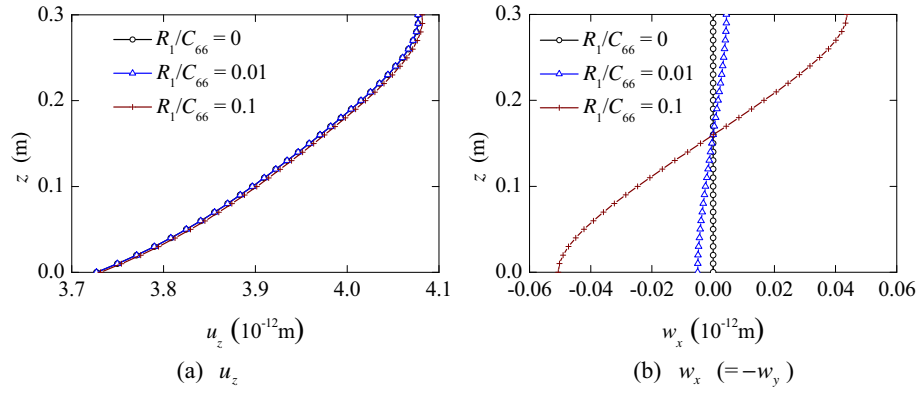


Fig. 9. Variation of the z-direction displacement components along z-direction of the plate for Case 1 under different coupling constant R_1 .

$$\bar{H}_k = 0. \quad (39)$$

By using Eq. (37) to process crystal layers as the “special QC” layers, the interface boundary condition in Eq. (39) can be very closely approximated. That is, the continuity conditions for z-direction phason traction forces along the interfaces in Eq. (7) can be satisfied. Therefore, for a multilayered plate containing both QC and crystal layers, using the solution in Eq. (35), the continuity conditions along the interfaces in Eq. (7) and the boundary conditions on its top or bottom surface, the phason physical quantities and phason stresses can be accurately obtained. We should further point out that, for crystal layers, since the phason elastic constants are very close to zero (relative to those in QC layers), the phason stress field in the crystal layer is also close to zero. As for the displacements in phason field, they should be zero or very close to zero in crystal layers. Since phason displacement represents the local rearrangement of the atoms in the unit cell, there is no

physical meaning at all for it in the crystal layer and one can simply set it to zero.

In conclusion, the general solution in Eq. (35) and the interface continuity conditions in Eq. (7) can be used to solve the problems of multilayered QC and crystal plates. In Section 5, a multilayered plate containing both QC and crystal will be particularly investigated.

4. Propagator method and solution for layered plates

By virtue of the general solution in Eq. (35), the interface continuity conditions in Eq. (7) and the boundary conditions on the top and bottom surfaces, the exact closed-form solution can be obtained for the multilayered QC plate shown in Fig. 1. To easily deal with a plate with relatively large numbers of layers, the propagator matrix method will be employed (Pan, 1997a).

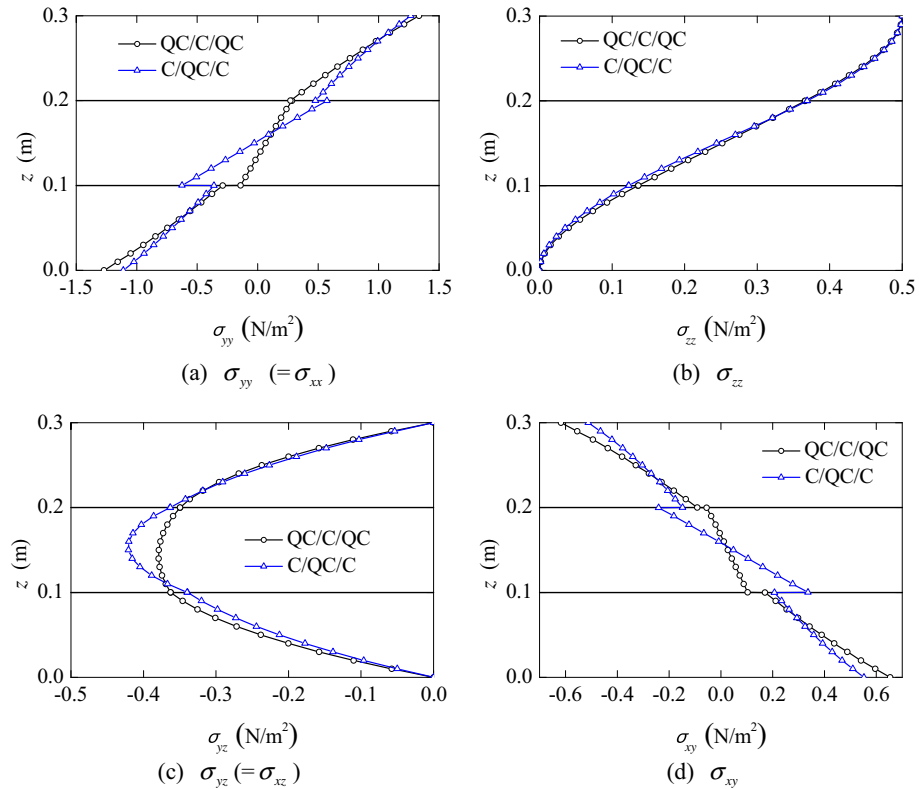


Fig. 10. Variation of the stress components in phason field along z-direction of the sandwich plate of Case 2 with $R_1/C_{66} = 0.01$ in the QC layer.

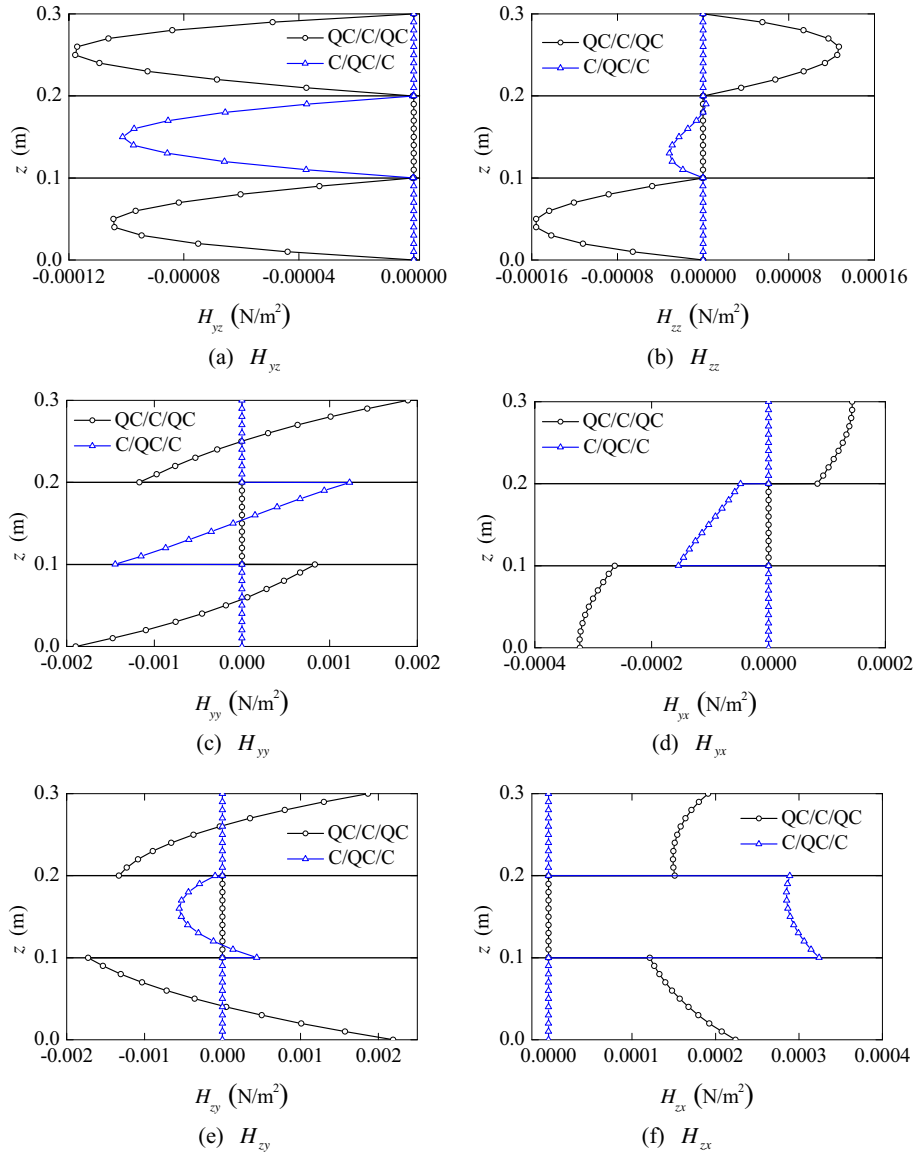


Fig. 11. Variation of the stress components in phason field along z -direction of the sandwich plate of Case 2 with $R_1/C_{66} = 0.01$ in the QC layer.

From Eq. (35), it can be seen that the constant column matrices \mathbf{K}_1 and \mathbf{K}_2 for layer j can be solved as follows:

$$\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}_j = \langle e^{s^*(z-z_j)} \rangle^{-1} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_z, \quad (40)$$

where the subscript j indicates layer j and s^* are the eigenvalues of layer j , and $z_j \leq z \leq z_{j+1}$. Letting z be equal to z_j and z_{j+1} , the column matrices, in the respective cases, are written as

$$\begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}_j = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=z_j} = \langle e^{s^* h_j} \rangle^{-1} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=z_{j+1}}, \quad (41)$$

where h_j is the thickness of layer j . From Eq. (41), the displacement \mathbf{u} and traction \mathbf{t} on the upper surface $z = z_{j+1}$ can be expressed in terms of those on the lower surface $z = z_j$ of layer j as

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=z_{j+1}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \langle e^{s^* h_j} \rangle \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=z_j}. \quad (42)$$

Assuming that both the displacement \mathbf{u} and traction \mathbf{t} are continuous across the interfaces, Eq. (42) can be applied repeatedly so that one can propagate the physical quantities from the bottom surface $z = 0$ to the top surface $z = H$ of the multilayered 2D QC plate. Therefore, we have

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=H} = \mathbf{P}_N(h_N) \cdots \mathbf{P}_j(h_j) \cdots \mathbf{P}_1(h_1) \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=0}, \quad (43)$$

where

$$\mathbf{P}_j(h_j) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \langle e^{s^* h_j} \rangle \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1}, \quad (j = 1, 2, \dots, N), \quad (44)$$

is defined as the propagating matrix or propagator matrix of layer j .

To calculate the inverse matrix in Eq. (44), the following simple relation in the pseudo-Stroh formalism can be used (Pan, 2001):

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{B}_2^t & \mathbf{A}_2^t \\ \mathbf{B}_1^t & -\mathbf{A}_1^t \end{bmatrix}, \quad (45)$$

where the matrices \mathbf{A}_1 , \mathbf{B}_1 , \mathbf{A}_2 and \mathbf{B}_2 are normalized according to

$$-\mathbf{B}_2^t \mathbf{A}_1 + \mathbf{A}_2^t \mathbf{B}_1 = \mathbf{I}, \quad (46)$$

with \mathbf{I} being a 5×5 unit matrix.

Eq. (43) is a very simple, yet powerful, matrix propagation relation. For given boundary conditions, the unknowns involved can be directly solved. As an example, we assume that a z -direction traction component is applied on the top surface of the plate such as

$$\sigma_{zz} = \sigma_0 \sin px \sin qy, \quad (47)$$

which may be one of the terms in the double Fourier series solution for a general loading case (uniform or point loading). All other traction components on the top and bottom surfaces of the plate are assumed to be zero. Thus, Eq. (43) is simplified to

$$\begin{pmatrix} \mathbf{u}(H) \\ \mathbf{t}(H) \end{pmatrix} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \mathbf{t}(0) \end{pmatrix}, \quad (48)$$

where \mathbf{C}_1 , \mathbf{C}_2 , \mathbf{C}_3 and \mathbf{C}_4 are the components of the product of the propagator matrices in Eq. (43), and $\mathbf{t}(H)$ is the given traction boundary condition on the top surface, i.e.

$$\mathbf{t}(H) = \{0, 0, \sigma_0 \sin px \sin qy, 0, 0\}^t. \quad (49)$$

Substitution Eq. (49) into Eq. (48) yields the unknown displacements at the bottom and top surfaces as

$$\mathbf{u}(0) = \mathbf{C}_3^{-1} \mathbf{t}(H), \quad \mathbf{u}(H) = \mathbf{C}_1 \mathbf{C}_3^{-1} \mathbf{t}(H). \quad (50)$$

Thus, the solution for the displacement and traction vectors at any depth $z_j \leq z \leq z_{j+1}$ is

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_z = \mathbf{P}_j(z - z_{j-1}) \mathbf{P}_{j-1}(h_{j-1}) \dots \mathbf{P}_2(h_2) \mathbf{P}_1(h_1) \begin{pmatrix} \mathbf{u} \\ \mathbf{t} \end{pmatrix}_{z=0}. \quad (51)$$

With the solved displacement and traction vectors at any given depth, the other stress components can be evaluated.

Similar exact closed-form solutions for various other boundary conditions can also be simply obtained. The exact closed-form solution for a multilayered rectangular 2D decagonal QC plate derived in this section is suitable for the four cases mentioned in Section 3. In the next section, we apply our solution to investigate the response of 2D QC plates under surface loadings.

5. Numerical examples

The first example is a homogeneous plate composed of a 2D decagonal QC with the three orientation cases as described in Section 3 (Case 1–3); the second example also considers a homogeneous plate with material orientation in Case 1 under different values of the coupling constant R_1 ; the third example is a sandwich

plate made of a 2D QC and a crystal with two stacking sequences based on the orientation Case 2. For the three examples, the same traction boundary condition is applied on the top of the plates by Eq. (49) with $n = m = 1$ and amplitude $\sigma_0 = 1 \text{ N/m}^2$, while on the top and bottom surfaces all other traction components are zero. To show the response of the plate in the thickness direction under the top surface loading, the horizontal coordinates are fixed at $(x, y) = (0.75L_x, 0.75L_y)$.

Example 1. Consider a square homogeneous plate made of a 2D decagonal QC with $L_x = L_y = 1 \text{ m}$ and $H = 0.3 \text{ m}$. According to Fan (2013), the material property constants for this 2D hexagonal QC are given as

$$\begin{aligned} C_{11} &= 23.433 \times 10^{10} \text{ N/m}^2, & C_{12} &= 5.741 \times 10^{10} \text{ N/m}^2, & C_{13} &= 6.663 \times 10^{10} \text{ N/m}^2, \\ C_{33} &= 23.222 \times 10^{10} \text{ N/m}^2, & C_{44} &= 7.019 \times 10^{10} \text{ N/m}^2, & C_{66} &= (C_{11} - C_{12})/2 = 8.846 \times 10^{10} \text{ N/m}^2, \\ R_1 &= 8.846 \times 10^8 \text{ N/m}^2, & K_1 &= 12.2 \times 10^{10} \text{ N/m}^2, \\ K_2 &= 2.4 \times 10^{10} \text{ N/m}^2, & K_4 &= 1.2 \times 10^{10} \text{ N/m}^2. \end{aligned}$$

Figs. 5 and 6 show respectively the variations of the stress components in the phonon and phason fields along z -direction in the homogeneous plate for the three orientation Cases of the structures. From the plots of the z -direction stresses, as shown in Fig. 5(c), (d), (f), and Fig. 6(e), (f) and (i), it can be seen that the values on the top and bottom surfaces satisfy the traction boundary conditions expressed in Eq. (49). This also partially verifies the correctness of the derived solution. From these figures, we observed clearly that the magnitude of the stress components in phonon field is much larger than that in phason field and that different orientations (different Cases) can substantially influence the distribution of the stress components in the phason field. The stresses shown in Figs. 5 and 6 further display the following characteristics:

- (1) In Case 1, the equivalent relations include: $\sigma_{xx} = \sigma_{yy}$, $\sigma_{xz} = \sigma_{yz}$, $H_{xx} = -H_{yy}$, $H_{xy} = -H_{yx}$ and $H_{xz} = -H_{yz}$.
- (2) In Cases 2 and 3, the equivalent relations are: $(\sigma_{xx})_{\text{Case 3}} = (\sigma_{yy})_{\text{Case 2}}$, $(\sigma_{xz})_{\text{Case 3}} = (\sigma_{yz})_{\text{Case 2}}$, $(\sigma_{xy})_{\text{Case 3}} = (\sigma_{yx})_{\text{Case 2}}$, $(H_{xx})_{\text{Case 3}} = -(H_{yy})_{\text{Case 2}}$, $(H_{xy})_{\text{Case 3}} = -(H_{yx})_{\text{Case 2}}$, $(H_{xz})_{\text{Case 3}} = -(H_{zx})_{\text{Case 2}}$, and $(H_{zz})_{\text{Case 3}} = -(H_{zz})_{\text{Case 2}}$. The relations still remain valid even if the subscripts “Case 2” and “Case 3” are interchanged.

Fig. 7 shows the variation of the displacement components in the phonon and phason fields along z -direction in the plate. Similarly, that the magnitude of the displacement components in phonon field is much larger than that in phason field and that different orientations (different Cases) can substantially influence the distribution of the displacement components in phason field. The displacements have the following characteristics:

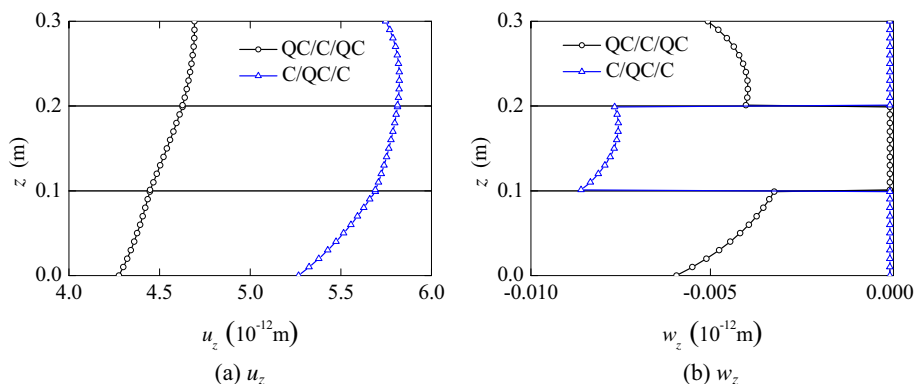


Fig. 12. Variation of u_z and w_z along z -direction of the sandwich plate of Case 2 with $R_1/C_{66} = 0.01$ in the QC layer.

- (1) In **Case 1**, equivalent relations include: $u_x = u_y$, and $w_x = -w_y$.
- (2) In **Cases 2 and 3**, there are equivalent relations: $(u_x)_{\text{Case 3}} = (u_y)_{\text{Case 2}}$, $(u_z)_{\text{Case 3}} = (u_z)_{\text{Case 2}}$, $(w_x)_{\text{Case 3}} = -(w_y)_{\text{Case 2}}$, and $(w_z)_{\text{Case 3}} = -(w_z)_{\text{Case 2}}$. By exchanging the subscripts “**Case 2**” and “**Case 3**” in the relations, they still hold.

Example 2. To analyze the coupling effect between the phonon and phason fields, we consider **Case 1** under different values of the coupling constant R_1 , which are $R_1/C_{66} = 0, 0.01$ and 0.1 . The geometry and the stress boundary conditions of the plate are the same as those in **Example 1**.

Figs. 8 and 9 show respectively the variations of the stress and displacement components in the phonon and phason fields along z -direction in the plate. It can be seen that the values of the coupling constant R_1 have significant influence on the displacement and stress components in phason field, while very weak influence on those components in phonon field.

Example 3. We now consider a sandwich plate made of the 2D QC of orientation **Case 2** with the coupling constant $R_1/C_{66} = 0.01$ and a crystal also in orientation **Case 2**. The geometry and the stress boundary conditions of the plate are also the same with those listed in **Example 1**. The three layers have equal thickness of 0.1 m. Two stacking sequences, QC/crystal/QC (called QC/C/QC) and crystal/QC/crystal (called C/QC/C), of the sandwich plate are investigated.

The material coefficients for the crystal are obtained by Lee and Jiang (1996) as

$$C_{11} = 16.6 \times 10^{10} \text{ N/m}^2, \quad C_{12} = 7.7 \times 10^{10} \text{ N/m}^2, \quad C_{13} = 7.8 \times 10^{10} \text{ N/m}^2, \\ C_{33} = 16.2 \times 10^{10} \text{ N/m}^2, \quad C_{44} = 4.3 \times 10^{10} \text{ N/m}^2, \\ C_{66} = (C_{11} - C_{12})/2 = 4.45 \times 10^{10} \text{ N/m}^2.$$

The phason elastic constants of the crystal are assumed using Eq. (37). In other words, in our calculation, we let, in the crystal layer, $R_1 = 0$ and a very small value for K_l ($l = 1, 2, 4$) which is about 10^{-10} of the corresponding K_l value in QC layer.

Figs. 10 and 11 show respectively the variation of the stress components in the phonon and phason fields along z -direction in the sandwich plate. From Figs. 10(b), (c), and 11(a), (b), it can be seen that the values of the traction components in Eq. (21) on the top and bottom surfaces satisfy the boundary conditions in Eq. (49). It is clear that the top surface loading produces quite different responses in these two structures, demonstrating the significant role played by the material stacking sequences. That the phason stresses in Fig. 11 are zero in crystal layers manifests the correctness of our processing method for the crystal layers. The two figures also show that the (in-plane) stress components in Eq. (23) are discontinuous across the interfaces and are nonzero on the bottom and top surfaces, while the traction components in Eq. (21) are continuous across the interfaces. These stress components are approximately either symmetric or antisymmetric about the middle plane.

Fig. 12 shows the variation of displacement components u_z and w_z along z -direction. It is clear that, across the interfaces, while the displacement in the phonon field is continuous, the displacement in the phason field is not. This feature on the phason displacement is consistent with and closely related to the fact that the phason displacement field corresponds to the local atomic rearrangement of unit cells.

6. Conclusions

Utilizing the powerful pseudo-Stroh formalism, we have derived an exact closed-form solution for a simply supported and

multilayered 2D decagonal QC plate under surface loadings. Based on the different relations between the periodic direction and the coordinate system of the plate, three internal structure cases for the QC layer are considered. The propagator matrix method is also introduced to efficiently and accurately treat the multilayered structures. A multilayered plate containing both QC layers and crystal layers is investigated in detail. The final exact closed-form solution has a concise and elegant expression.

A homogeneous plate with different internal structures under a surface loading on the top of the plate is numerical investigated. It can be seen that the internal structures have distinguishable influence on all physical quantities, especially on the physical quantities in phason field. Under different coupling constants, a homogeneous plate with the internal structure in **Case 1** is also studied numerically under the same boundary conditions. The results show that the coupling constant strongly influences the stress and displacement components in phason field but only weakly influences those in phonon field. These results are closely related to the loading condition of the problems. From the numerical example of a sandwich plate made of a 2D QC and a crystal with two stacking sequences, it is observed that the stacking sequences can substantially influence all physical quantities especially at the interface. The exact closed-form solution of this paper should be of interest to the design of the 2D QC homogeneous and laminated plates. The results can also be employed to verify the accuracy of the solution by numerical methods, such as the finite element and difference methods, when analyzing laminated composites made of QCs.

Acknowledgements

The work is supported by the National Natural Science Foundation of China (No. 11172319), Chinese Universities Scientific Fund (No. 2011JS046 and No. 2013BH008), Opening Fund of State Key Laboratory of Nonlinear Mechanics, Program for New Century Excellent Talents in University (No. NCET-13-0552), and National Science Foundation for Post-doctoral Scientists of China (No. 2013M541086).

References

- Altay, G., Dökmeci, M.C., 2012. On the fundamental equations of piezoelectricity of quasicrystal media. *Int. J. Solids Struct.* 49, 3255–3262.
- Bak, P., 1985a. Phenomenological theory of icosahedral incommensurate (“Quasiperiodic”) order in Mn–Al alloys. *Phys. Rev. Lett.* 54, 1517–1519.
- Bak, P., 1985b. Symmetry, stability, and elastic properties of icosahedral incommensurate crystals. *Phys. Rev. B* 32, 5764–5772.
- Balyshev, V.N., King, D.J., Khramov, A.N., Kasten, L.S., Donley, M.S., 2004. Investigation of quaternary Al-based quasicrystal thin films for corrosion protection. *Thin Solid Films* 447–448, 558–563.
- Ding, D.H., Yang, W.G., Hu, C.Z., Wang, R.H., 1993. Generalized elasticity theory of quasicrystals. *Phys. Rev. B* 48, 7003–7010.
- Fan, T.Y., 2011. *Mathematical Elasticity of Quasicrystals and Its Applications*. Springer, Heidelberg.
- Fan, T.Y., 2013. Mathematical theory and methods of mechanics of quasicrystalline materials. *Engineering* 5, 407–448.
- Fan, T.Y., Trebin, H.R., Messerschmidt, U., Mai, Y.W., 2004. Plastic flow coupled with a crack in some one- and two- dimensional quasicrystals. *J. Phys. Condens. Matter* 16, 5229–5240.
- Fan, T.Y., Xie, L.Y., Fan, L., Wang, Q.Z., 2011. Interface of quasicrystal and crystal. *Chin. Phys. B* 20, 076102.
- Hu, C.Z., Ding, R.H., Ding, D.H., 2000. Symmetry groups, physical property tensors, elasticity and dislocations in quasicrystals. *Rep. Prog. Phys.* 63, 1–39.
- Kausel, E., Roesset, J.M., 1981. Stiffness matrices for layered soils. *Bull. Seismol. Soc. Am.* 71, 1743–1761.
- Lee, J.S., Jiang, L.Z., 1996. Exact electroelastic analysis of piezoelectric laminae via state space approach. *Int. J. Solids Struct.* 33, 977–990.
- Levine, D., Steinhardt, P.J., 1984. Quasicrystals: a new class of ordered structures. *Phys. Rev. Lett.* 53, 2477–2480.
- Noor, A.K., Burton, W.S., 1990. Three-dimensional solutions for antisymmetrically laminated anisotropic plates. *ASME J. Appl. Mech.* 57, 182–188.
- Pan, E., 1997a. Static Green’s functions in multilayered half spaces. *Appl. Math. Model.* 21, 509–521.

- Pan, E., 1997b. A general boundary element analysis of 2-D linear elastic fracture mechanics. *Int. J. Fract.* 88, 41–59.
- Pan, E., 2001. Exact solution for simply supported and multilayered magneto-electro-elastic plates. *J. Appl. Mech.* 68, 608–618.
- Rochal, S.B., Lorman, V.L., 2002. Minimal model of the phonon–phason dynamics on icosahedral quasicrystals and its application for the problem of internal friction in the i-AlPdMn alloy. *Phys. Rev. B* 66, 144204-1–144204-9.
- Shechtman, D., Blech, I., Gratias, D., Cahn, J.W., 1984. Metallic phase with long-range orientational order and no translational symmetry. *Phys. Rev. Lett.* 53, 1951–1953.
- Shi, W.C., 2005. Conservation laws of decagonal quasicrystal in elastodynamics. *Eur. J. Mech. A. Solids* 24, 217–226.
- Sladek, J., Sladek, V., Pan, E., 2013. Bending analyses of 1D orthorhombic quasicrystal plates. *Int. J. Solids Struct.* 50, 3975–3983.
- Stroh, A.N., 1958. Dislocations and cracks in anisotropic elasticity. *Philos. Mag.* 3, 625–646.
- Zhou, W.M., Fan, T.Y., 2001. Plane elasticity problem of two-dimensional octagonal quasicrystals and crack problem. *Chin. Phys.* 10, 743–747.